ON NUMERICAL SCHEMES FOR SOLVING A NONLINEAR PDE WITH A MEAN CURVATURE TERM ON TRIANGULAR MESHES

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Mathematical Modelling and Numerical Analysis Will be set by the publisher
Modélisation Mathématique et Analyse Numérique

ON NUMERICAL SCHEMES FOR SOLVING A NONLINEAR PDE WITH A MEAN CURVATURE TERM ON TRIANGULAR MESHES. *

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Abstract. In this paper, we propose two numerical schemes for solving a nonlinear level set equation on unstructured triangulations. The consistency and the stability of these schemes are shown. In addition, we show how to compute nodewise fisrt and second order derivatives. An application example of curve construction using these approximations is provided to demonstrate their accuracy.

Résumé. Dans ce papier, nous proposons deux schémas numériques pour résoudre une équation non linéaire de type ligne de niveaux sur des maillages triangulaires. Nous montrons, de plus, comment calculer les dérivées premières et secondes aux nœuds du maillage. On établit la consistance et la stabilité de ces deux schémas et on fournit un exemple d'application à la construction d'une courbe utilisant ces approximations, pour démontrer leur efficacité.

1991 Mathematics Subject Classification. 65N06, 65N12, 65N22.

The dates will be set by the publisher.

1. Introduction

In [6], the authors applied the level set method to solve the classical problem of fitting a regular curve through a set of unorganized points. This challenging shape approximation issue has many potential applications in scientific computing, interface tracking, image processing/segmentation and biomedical simulations. However, it is mathematically an ill-posed problem, as there is usually no unique solution. Initially introduced for tracking moving interfaces, the level set method is used here to deform continuously an initial regular curve implicitly defined, until it fits at best through the given set of points. This method presents indeed two desirable features. At first, it can deal with arbitrary points sets, including noisy and non uniform data. Furthermore, it can handle topological changes easily during curve evolution. Such methods are known for developing singularities near regions of topological changes or where the gradient of the level set function vanishes, that yield inaccurate calculations of high order terms in the equations. To overcome this problem, essentially (weighted) non-oscillatory residual distribution schemes have been proposed on Cartesian grids [16, 18, 23] and ultimately extended to triangular meshes [1]. Nonetheless, the accurate and robust evaluation of curvature terms on unstructured triangulations remains important for solving this class of problems.

Keywords and phrases: analysis of a numerical scheme, unstructured mesh, level set method, mean curvature evolution, second order derivatives approximation, non linear PDE problem, curve reconstruction.

* *The first author is supported by a research grant from the Région Ile-de-France.*

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1.1. Overview of the level set approach

Given a point set V, the objective is to construct a closed curve Γ passing "through" all points of V. This is equivalent to say that every point x in V lies on, or is very close to Γ, *i.e.,* for ε small we have $d(x,\Gamma) \leq \varepsilon$, $\forall x \in V$, where $d(x,\Gamma) = \min_{\bar{x}\in\Gamma} \|x-\bar{x}\|$ denotes the distance function to the curve Γ. To this end, we consider a level set formulation of a curve evolution problem. The preliminary step consists in embedding the unknown curve Γ in the unit domain $\Omega = [0,1]^2$, and then to consider $\Gamma = \{x\,,\ u(t,x) = 0\}$ as the zero isocontour of a scalar function u defined on $\mathbb{R}\times\Omega$. The closed curve Γ allows to define a continuous level set function u, positive on one side of the curve and negative on the other side. Next, following [26], we introduce a general evolution $\Gamma(t)_{t\geq 0}$ of an initial curve $\Gamma(0)$, by representing $\Gamma(t)$, for all $t \geq 0$, as the zero level set of the auxiliary function u, precisely,

$$\Gamma(t) = \{x \in \Omega,\ u(t,x) = 0\}\,, \qquad \forall t \geq 0\,.$$

Formally, the function u, *i.e.,* the curve $\Gamma(t)$ and all the level sets, has to satisfy, for all $x \in \Gamma(t)$, the equation $u(t,x) = u(t,\Gamma(t)) = 0$. In our problem, the evolution of the curve $\Gamma(t)$ is purely geometric and thus the same velocity law is applied on each level set of the function u that will then follow the same equation and move along the normal direction to the level set function. Differentiating $u(t,x) = u(t,\Gamma(t)) = 0$, with respect to t yields the following equation:

$$\frac{du}{dt}(t,\Gamma(t,s)) = \frac{\partial u}{\partial t}(t,\Gamma(t,s)) + \frac{d\Gamma(t,s)}{dt}\cdot\nabla u(t,\Gamma(t,s)) = 0\,, \tag{1}$$

where s denotes the curvilinear abscissa and $\Gamma(t,s)$ any point along $\Gamma(t)$.

Level sets techniques have been extensively studied in front propagation and interface tracking problems. The mean curvature formulation was introduced by [22] for numerical purposes and rigorous explanations based on the notion of viscosity solutions have been provided by [8,17]. In our approach, the motion of the curve $\Gamma(t)$ and all the iso level sets is prescribed by the nonlinear equation and the initial condition:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \alpha d(x) + d(x)\kappa(u)(t,x) = \alpha d(x) + d(x)\left(\nabla\cdot\left(\dfrac{\nabla u}{|\nabla u|}\right)\right)(t,x)\,, \\ u(0,\cdot) = u_0 \end{cases} \tag{2}$$

that relates the distance d to the points set and the curvature κ of the level sets of u, α being a positive scaling parameter. The given data is the initial front $\Gamma(0)$ which is supposed to be the boundary of an open set and no more regularity is assumed on this curve. The function u_0 is any uniformly continuous function such that:

$$\{u_0 = 0\} = \Gamma(0)\,, \quad \{u_0 > 0\} = \Omega_+\,, \quad \text{and} \quad \{u_0 < 0\} = \Omega\backslash(\Gamma(0)\cup\Omega_+)\,,$$

for instance the signed distance to $\Gamma(0)$ which is positive in Ω_+. The fundamental theorem of level set method [12] states that the general evolution by mean curvature $\Gamma(t)_{t\geq 0}$ of the couple $(\Gamma(0),\Omega_+)$ is well-defined in the sense that $\Gamma(t)$ depends only on the initial data $(\Gamma(0),\Omega_+)$, and not on u_0. Choosing another function u_0 may affect the solution but not its zero level set.

Using the same approaches and relying on assumptions described in the papers [3,15], the existence and uniqueness of a viscosity solution for the Equation (2) can be established. Details about this proof can be read in [7].

The formulation (2) shows a good balance between the attraction term αd and the scaled surface tension term $d\kappa$. The coefficient α can be chosen accordingly to obtain the desired regularity related to the curvature term. Furthermore, it has numerous advantages, among which two are especially interesting here: it can be computed numerically [25] and it is defined for all times regardless of the regularity of the initial curve. In addition, geometric quantities such as normal and curvature can be calculated directly from the level set function u, but require accurate and robust numerical schemes.

1.2. **Problem statement: numerical difficulties**

In most problems that require the discretization of the computational domain, the accuracy of the solution and its quality (e.g. its overall regularity) are intrinsically related to the quality of the mesh as well as to its adaptation to the problem at hand. With the type of discontinuities developing in level set methods, standard centered finite difference schemes for computing the curvature term become inaccurate and unstable, *i.e.,* lead to spikes in the curvature error. Refining locally the Cartesian mesh may only postpone the problem without eliminating it [20]. Furthermore, the mesh cannot be refined indefinitely to capture vanishing gradients as it will lead to restrictive time steps with explicit schemes.

On the other hand, the finite element method has become eminently popular in engineering applications as it involves a variational formulation of the problem and looks for solutions in suitable functional spaces. It offers the flexibility of dealing with an unstructured triangular mesh of the domain, possibly adapted to the geometry of the domain boundary as well as to the solution variation. However, the approximation of second order derivatives is related to the degree of the polynomials associated with the shape functions. Obtaining an accurate nodal value (at the mesh vertices) of the second order terms in Equation (2) would require using at least third degree polynomials, thus leading to a substantial increase of the number of degrees of freedom in our problem. In addition, non oscillatory schemes are sometimes relatively tedious to implement on triangulations (see [1] for a survey).

Finally, as pointed out by [20], using a local level set function is more robust in calculating the curvature than directly differentiating an interpolating function like a spline for instance. To our knowledge, very few works have dealt with finite difference schemes on unstructured meshes [5, 21] and even less in the context of level set techniques. In this paper, we propose an alternative for computing a curvature value at each mesh vertex without having to resort on high order interpolation schemes. Moreover, this approach is easy to implement.

1.3. **Numerical resolution**

The first order differential operator with respect to the time variable t in Equation (2) can be discretized using a classical upwind finite difference scheme with a time step Δt. We introduce the notations $u_i^n \approx u(x_i, t^n)$ and $d_i \approx d(x_i)$, where $d(x_i)$ is the distance function to the points of V. We propose two schemes for solving the Equation (2):

- an explicit scheme, written as:

$$u_i^{n+1} = u_i^n + d_i\,\Delta t\,\left(\left(\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\right)_i^n + \alpha\right) \tag{3}$$

- a semi-implicit scheme, written as:

$$u_i^{n+1} - d_i\,\Delta t\,\left(\nabla\cdot\left(\frac{\nabla u^{n+1}}{|\nabla u^n|}\right)\right)_i = d_i\,\Delta t\,\alpha + u_i^n. \tag{4}$$

In Section 2, we introduce a technique for computing the approximation of the first and second order spatial derivatives on triangulations. Section 3 and 4, analyze successively the consistency and the stability of the proposed schemes and, Section 5, we provide an application example of curve reconstruction to emphasize its efficiency.

2. Approximation of the spatial derivatives

As mentioned previously, our model Equation (2) involves the first (gradient) and the second (mean curvature) order derivatives of the level set function u. Hence, it is important for the efficiency and the robustness of the numerical method to compute these derivatives as accurately as possible. With finite difference/element approximations, the level set function u is defined at the grid points, and so are the spatial derivatives. Hence, we would like to define these derivatives at the vertices of a triangulation. Next, we propose a method for evaluating the gradient and the mean curvature at each node of a given triangulation T_h.

2.1. **Gradient approximation**

Introducing the barycentric coordinates ω_{Ki_2} of a vertex x_{i_2} of each triangle $K \in T_h$ leads to write:

$$u|_K = \sum_{i_2 \in K} u_{i_2}\, \omega_{Ki_2}\,, \qquad \text{and} \qquad (\nabla u)|_K = \sum_{i_2 \in K} u_{i_2}\, \nabla\omega_{Ki_2}\,. \tag{5}$$

Then, for each vertex i in the triangulation T_h, the discrete gradient is defined as:

$$(\nabla u)_i = \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K (\nabla u)|_K}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} = \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K \sum_{i_2\in K} u_{i_2}\nabla\omega_{Ki_2}}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} \tag{6}$$

where A_K is the area of triangle K, $\mathcal{B}_i$ is the support of i, *i.e.,* the set of triangles containing vertex i.

2.2. **Mean curvature approximation**

Once the gradient approximation has been obtained at each mesh vertex, the same procedure can be applied to each component of the gradient vector to supply a mean curvature value at each vertex as well. The previous formulas allows us to write: $(\nabla(\nabla u))|_K = \displaystyle\sum_{i\in K} (\nabla u)_i (\nabla\omega_{Ki})^T$, where we considered the raw vector $(\nabla\omega_{Ki})^T$, in order to calculate the product. With this definition, $(\nabla(\nabla u))|_K$ is a square matrix. The aim is to obtain the value $(\nabla(\nabla u))_i$ at each point i. Hence, using the previous formulas (5) and (6), we compute successively the coefficients of the Hessian matrix:

$$(\nabla(\nabla u))_i = \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K\, \nabla(\nabla u)|_K}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} = \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K \left(\sum_{i_2\in K} \left(\frac{\displaystyle\sum_{L\in\mathcal{B}_{i_2}} A_L \left(\sum_{i_3\in L} u_{i_3}\nabla\omega_{Li_3} \right)}{\displaystyle\sum_{L\in\mathcal{B}_{i_2}} A_L} \right) (\nabla\omega_{Ki_2})^T \right)}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} \tag{7}$$

and the trace of the Hessian matrix:

$$(\nabla\cdot(\nabla u))_i = \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K \left(\sum_{i_2\in K} \left(\frac{\displaystyle\sum_{L\in\mathcal{B}_{i_2}} A_L \left(\sum_{i_3\in L} u_{i_3} \langle \nabla\omega_{Li_3}\,,\, \nabla\omega_{Ki_2} \rangle \right)}{\displaystyle\sum_{L\in\mathcal{B}_{i_2}} A_L} \right) \right)}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} \tag{8}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidian scalar product. Futhermore, the local mean curvature at a mesh vertex i is then defined as:

$$\kappa_i = \left(\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\right)_i = \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K \left(\sum_{i_2\in K} \left(\frac{\displaystyle\sum_{L\in\mathcal{B}_{i_2}} A_L \left(\sum_{i_3\in L} u_{i_3} \langle \nabla\omega_{Li_3}\,,\, \nabla\omega_{Ki_2} \rangle \right)}{\displaystyle |\nabla u|_{i_2} \sum_{L\in\mathcal{B}_{i_2}} A_L} \right) \right)}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K}\,. \tag{9}$$

Notice that in this expression, the set $\mathcal{B}_i$ denotes all the triangles K containing the mesh vertex i and the set $\mathcal{B}_{i_2}$ caracterizes all the triangles L containing i_2.

From the numerical point of view, these two approximations are relatively easy to implement. The sole difficulty is related to the fast identification of all triangles in the sets $\mathcal{B}_i$, for all i. This can be achieved using appropriate data structures, especially since T_h is kept unchanged during all the evolution of the curve [13].

3. Consistency

In this section, we establish the consistency of the proposed schemes with respect to the model Equation (2). To this end, we start by considering the Laplacian case, that leads to reduce the curvature term. We also assume that the underlying mesh is a regular Cartesian grid. These two hypothesis will allow a better understanding of the proof. In a second stage, we will analyze the general case, where $\kappa(u) = \nabla \cdot \left(\dfrac{\nabla u}{|\nabla u|}\right)$.

3.1. **The Laplacian case**

We consider a Cartesian grid T_h of generic size h and we denote K a square cell of T_h, (i,j) being the point at wich we calculate the Laplacian. Let $\mathcal{B}_{i,j}$ be the ball of the point (i,j), *i.e.,* the set of the cells containing (i,j). By analogy with the formulas (5) and (6), we write the expressions of the gradient at (i,j):

$$(\nabla u)|_K = \sum_{(i,j)\in K} u_{i,j} \nabla w_{K,i,j}, \qquad (\nabla u)_{i,j} = \frac{\displaystyle\sum_{K\in\mathcal{B}_{i,j}} A_K (\nabla u)|_K}{\displaystyle\sum_{K\in\mathcal{B}_{i,j}} A_K} = \frac{\displaystyle\sum_{K\in\mathcal{B}_{i,j}} \sum_{(i_2,j_2)\in K} u_{i_2,j_2} \nabla\omega_{K,i_2,j_2}}{4}$$

where the value 4 at the denominator of the last fraction corresponds to the number of cells sharing (i,j). By extension, the curvature evaluation yields:

$$\begin{aligned}
(\nabla\cdot(\nabla u))_{i,j} = (\Delta u)_{i,j} &= \frac{1}{4}\,\frac{\displaystyle\sum_{K\in\mathcal{B}_{i,j}} A_K \left(\sum_{(i_2,j_2)\in K} \left(\frac{\displaystyle\sum_{L\in\mathcal{B}_{i_2,j_2}} A_L \left(\sum_{(i_3,j_3)\in L} u_{i_3,j_3} \langle \nabla\omega_{L,i_3,j_3}, \nabla\omega_{K,i_2,j_2}\rangle \right)}{\displaystyle\sum_{L\in\mathcal{B}_{i_2,j_2}} A_L} \right)\right)}{\displaystyle\sum_{K\in\mathcal{B}_{i,j}} A_K} \\
&= \frac{1}{4}\,\frac{\displaystyle\sum_{K\in\mathcal{B}_{i,j}} \sum_{(i_2,j_2)\in K} \left(\sum_{L\in\mathcal{B}_{i_2,j_2}} \sum_{(i_3,j_3)\in L} u_{i_3,j_3} \langle \nabla\omega_{L,i_3,j_3}, \nabla\omega_{K,i_2,j_2}\rangle \right)}{16}.
\end{aligned} \tag{10}$$

Here, the coefficient $\frac{1}{4}$ in the first equation comes from the calculation of the term $\langle\nabla\omega, \nabla\omega\rangle$ in square cells. Similarly, the number of cells in $\mathcal{B}_{i,j}$ $\times$ the number of squares in $\mathcal{B}_{i_2,j_2}$ appears also in the denominator. The stencil of the Laplace operator for the scheme considered is depicted in Figure 1.

After performing all the computations, the expression of the Laplacian at (i,j) finally becomes:

$$\begin{aligned}
(\Delta u)_{i,j} = -\frac{1}{64h^2}\big(&24u_{i,j} + 8u_{i+1,j} + 8u_{i,j+1} + 8u_{i-1,j} + 8u_{i,j-1} \\
&- 2u_{i+2,j-2} - 2u_{i+2,j+2} - 2u_{i-2,j+2} - 2u_{i-2,j-2} \\
&- 4u_{i-1,j-2} - 4u_{i,j-2} - 4u_{i+1,j-2} - 4u_{i+2,j-1} - 4u_{i+2,j} - 4u_{i+2,j+1} \\
&- 4u_{i+1,j+2} - 4u_{i,j+2} - 4u_{i-1,j+2} - 4u_{i-2,j+1} - 4u_{i-2,j} - 4u_{i-2,j-1}\big).
\end{aligned} \tag{11}$$

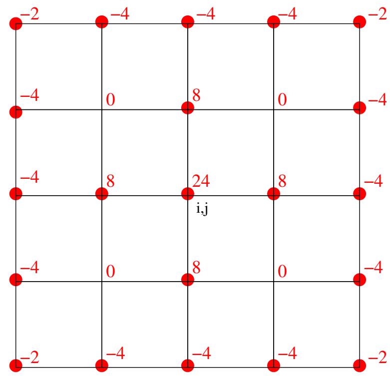

FIGURE 1. Stencil of the Laplace operator at (i, j) and related coefficients.

It remains to be proved that this expression is consistent with the Laplace operator.

In one dimension of space, or if we assume that $u_{i,j}$ depends only on j, summing each coefficient of each column in the previous expression (11) leads to write:

$$\frac{1}{64}\left(\frac{16u_{i+2} - 32u_i + 16u_{i-2}}{h^2}\right) = \frac{u_{i+2} - 2u_i + u_{i-2}}{(2h)^2} \sim \Delta u.$$

We use classical Taylor expansions up to the order four and we are able to show that the approximation is consistent at the order 2 with the Laplace operator in this special case. Indeed, we have:
Suppose $u \in C^4$, there exists $y \in [x_i, x_{i+2}]$ such that:

$$u(t^n, x_{i+2}) = u(t^n, x_i) + 2h\,\partial_x u(t^n, x_i) + \frac{(2h)^2}{2}\,\partial^2_{x,x} u(t^n, x_i) + \frac{(2h)^3}{6}\,\partial^3_{x,x,x} u(t^n, x_i) + \frac{(2h)^4}{24}\,\partial^4_{x,x,x,x} u(t^n, y).$$

There exists also $z \in [x_{i-2}, x_i]$ such that :

$$u(t^n, x_{i-2}) = u(t^n, x_i) - 2h\,\partial_x u(t^n, x_i) + \frac{(2h)^2}{2}\,\partial^2_{x,x} u(t^n, x_i) - \frac{(2h)^3}{6}\,\partial^3_{x,x,x} u(t^n, x_i) + \frac{(2h)^4}{24}\,\partial^4_{x,x,x,x} u(t^n, z).$$

Then, we have :

$$\begin{aligned}\frac{u(t^n, x_{i+2}) - 2u(t^n, x_i) + u(t^n, x_{i-2})}{(2h)^2} &= \frac{1}{4h^2}\left(4h^2\,\partial^2_{x,x} u(t^n, x_i) + \frac{2h^4}{3}\,\partial^4_{x,x,x,x} u(t^n, y) + \frac{2h^4}{3}\,\partial^4_{x,x,x,x} u(t^n, z)\right)\\ &= \Delta u(t^n, x_i) + \frac{h^2}{12}\left(\partial^4_{x,x,x,x} u(t^n, y) + \partial^4_{x,x,x,x} u(t^n, z)\right).\end{aligned}$$

If we pose $\varepsilon^n_j(u)$ the consistency error then we deduce:

$$\varepsilon^n_j(u) = \frac{u(t^n, x_{i+2}) - 2u(t^n, x_i) + u(t^n, x_{i-2})}{(2h)^2} - \Delta u(t^n, x_i) \le C\,h^2,$$

where C is a constant independent of h and depends only on the four order space derivatives.

In two dimensions of space, the following values inside the bracket signs correspond to the coefficients of the stencil (and shall not be taken as a matrix):

$$\frac{1}{64h^2}\begin{bmatrix} 2 & 4 & 4 & 4 & 2\\ 4 & 0 & -8 & 0 & 4\\ 4 & -8 & -24 & -8 & 4\\ 4 & 0 & -8 & 0 & 4\\ 2 & 4 & 4 & 4 & 2 \end{bmatrix}.$$

We split the analysis into three steps and we successively consider

- the principal cross in the stencil:

$$\frac{1}{64h^2}\begin{bmatrix} & & 4 & & \\ & & 0 & & \\ 4 & 0 & -16 & 0 & 4 \\ & & 0 & & \\ & & 4 & & \end{bmatrix}$$

  that leads to the expression:

$$\frac{1}{64}\left(\frac{4u_{i,j-2}-8u_{i,j}+4u_{i,j+2}}{h^2}+\frac{4u_{i+2,j}-8u_{i,j}+4u_{i-2,j}}{h^2}\right)\sim\frac{1}{4}\Delta u, \tag{12}$$

- then, the terms:

$$\frac{1}{64h^2}\begin{bmatrix} & 4 & & 4 & \\ 4 & 0 & -8 & 0 & 4 \\ & -8 & & -8 & \\ 4 & 0 & -8 & 0 & 4 \\ & 4 & & 4 & \end{bmatrix}$$

  that give:

$$\begin{aligned}&\frac{1}{64}\left(\frac{4u_{i-2,j+1}-8u_{i,j+1}+4u_{i+2,j+1}}{h^2}+\frac{4u_{i+1,j-2}-8u_{i+1,j}+4u_{i+1,j+2}}{h^2}\right)\\ +&\frac{1}{64}\left(\frac{4u_{i-2,j-1}-8u_{i,j-1}+4u_{i+2,j-1}}{h^2}+\frac{4u_{i-1,j-2}-8u_{i-1,j}+4u_{i-1,j+2}}{h^2}\right)\sim\frac{1}{2}\Delta u,\end{aligned} \tag{13}$$

- and the remaining terms in the stencil:

$$\frac{1}{64h^2}\begin{bmatrix} 2 & & & & 2 \\ & 0 & & 0 & \\ & & -8 & & \\ & 0 & & 0 & \\ 2 & & & & 2 \end{bmatrix}$$

  leading to:

$$\frac{1}{64}\left(\frac{2u_{i-2,j+2}-4u_{i,j}+2u_{i+2,j-2}}{h^2}+\frac{2u_{i-2,j-2}-4u_{i,j}+2u_{i+2,j+2}}{h^2}\right)\sim\frac{2(2\sqrt{2})^2}{64}\Delta u=\frac{1}{4}\Delta u. \tag{14}$$

By summing the expressions (12), (13) and (14), we can observe that the scheme is a consistent approximation of the Laplace operator, at the second order, using again Taylor expansions as previously shown.

3.2. The general case

In turn, we now consider the evaluation of the curvature formula $\kappa(u)=\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)$ on a Cartesian grid. According to the previous calculations, at each grid point (i,j), we can write the following equation:

$$\left(\nabla\cdot\left(\frac{\nabla u}{a}\right)\right)_{i,j}=\frac{1}{64}\sum_{K\in\mathcal{B}_{i,j}}\sum_{(i_2,j_2)\in K}\left(\frac{\displaystyle\sum_{L\in\mathcal{B}_{i_2,j_2}}\sum_{(i_3,j_3)\in L}u_{i_3,j_3}\langle\nabla\omega_{L,i_3,j_3},\nabla\omega_{K,i_2,j_2}\rangle}{a_{i_2,j_2}}\right)$$

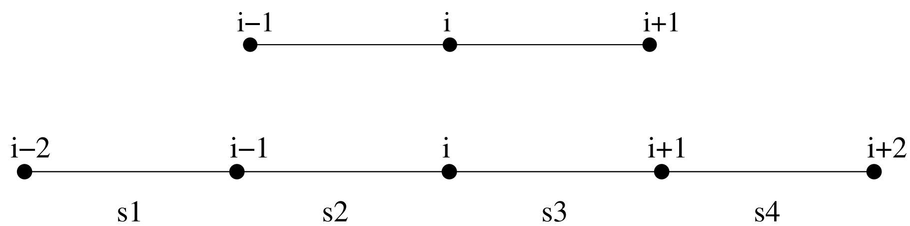

FIGURE 2. The one dimensional case: the classical stencil (top) vs. our stencil (bottom).

where $a = |\nabla u|$. We denote $c_{k,l}$ the coefficients of the points (k, l) in $L \in \mathcal{B}_{i_2}$ contributing to the value of the stencil of $\left(\nabla \cdot \left(\frac{\nabla u}{a}\right)\right)_{i,j}$. These terms can be expressed as:

$$c_{i,j} = \frac{4}{a_{i+1,j}} + \frac{4}{a_{i,j+1}} + \frac{4}{a_{i-1,j}} + \frac{4}{a_{i,j-1}} + \frac{2}{a_{i+1,j+1}} + \frac{2}{a_{i-1,j+1}} + \frac{2}{a_{i-1,j-1}} + \frac{2}{a_{i+1,j-1}},$$

$$c_{i-1,j-1} = 0, \qquad c_{i,j-1} = \frac{2}{a_{i-1,j}} + \frac{2}{a_{i-1,j-1}} + \frac{2}{a_{i+1,j-1}} + \frac{2}{a_{i+1,j}},$$

$$c_{i+1,j-1} = 0, \qquad c_{i+1,j} = \frac{2}{a_{i+1,j-1}} + \frac{2}{a_{i,j-1}} + \frac{2}{a_{i+1,j+1}} + \frac{2}{a_{i,j+1}},$$

$$c_{i+1,j+1} = 0, \qquad c_{i,j+1} = \frac{2}{a_{i+1,j}} + \frac{2}{a_{i-1,j}} + \frac{2}{a_{i+1,j+1}} + \frac{2}{a_{i-1,j+1}},$$

$$c_{i-1,j+1} = 0, \qquad c_{i-1,j} = \frac{2}{a_{i-1,j+1}} + \frac{2}{a_{i,j+1}} + \frac{2}{a_{i-1,j-1}} + \frac{2}{a_{i,j-1}},$$

$$c_{i-2,j-2} = \frac{-2}{a_{i-1,j-1}}, \qquad c_{i-1,j-2} = \frac{-2}{a_{i,j-1}} - \frac{2}{a_{i-1,j-1}}, \qquad c_{i,j-2} = \frac{-4}{a_{i,j-1}}, \qquad c_{i+1,j-2} = \frac{-2}{a_{i+1,j-1}} - \frac{2}{a_{i,j-1}},$$

$$c_{i+2,j-2} = \frac{-2}{a_{i+1,j-1}}, \qquad c_{i+2,j-1} = \frac{-2}{a_{i+1,j}} - \frac{2}{a_{i+1,j-1}}, \qquad c_{i+2,j} = \frac{-4}{a_{i+1,j}}, \qquad c_{i+2,j+1} = \frac{-2}{a_{i+1,j+1}} - \frac{2}{a_{i+1,j}},$$

$$c_{i+2,j+2} = \frac{-2}{a_{i+1,j+1}}, \qquad c_{i+1,j+2} = \frac{-2}{a_{i,j+1}} - \frac{2}{a_{i+1,j+1}}, \qquad c_{i,j+2} = \frac{-4}{a_{i,j+1}}, \qquad c_{i-1,j+2} = \frac{-2}{a_{i-1,j+1}} - \frac{2}{a_{i,j+1}},$$

$$c_{i-2,j+2} = \frac{-2}{a_{i-1,j+1}}, \qquad c_{i-2,j+1} = \frac{-2}{a_{i-1,j}} - \frac{2}{a_{i-1,j+1}}, \qquad c_{i-2,j} = \frac{-4}{a_{i-1,j}}, \qquad c_{i-2,j-1} = \frac{-2}{a_{i-1,j-1}} - \frac{2}{a_{i-1,j}}.$$

In one dimension of space, we recall that the classical scheme involves only the grid points $i-1$, i, $i+1$ (Figure 2, top) and then the curvature formula becomes:

$$\delta_x \left(\frac{\delta_x u}{a}\right)_i = \frac{\left(\frac{\delta_x u}{a}\right)_{i+\frac{1}{2}} - \left(\frac{\delta_x u}{a}\right)_{i-\frac{1}{2}}}{\Delta x} = \frac{1}{h^2}\left(\frac{u_{i+1}-u_i}{a_{i+\frac{1}{2}}} - \frac{u_i - u_{i-1}}{a_{i-\frac{1}{2}}}\right).$$

If the grid points $i + \frac{1}{2}$ and $i - \frac{1}{2}$ are not defined, we could use instead an average value, $a_{i+\frac{1}{2}} = \frac{1}{2}(a_{i+1} + a_i)$.

With the proposed scheme, the curvature expression reads as follows:

$$\begin{aligned}\delta_x \left(\frac{\delta_x u}{a}\right)_i &= \frac{u_{i+2}\,\omega'_{s_4,i+2}\,\omega'_{s_3,i+1}}{a_{i+1}} + \frac{u_i\,\omega'_{s_3,i}\,\omega'_{s_3,i+1}}{a_{i+1}} + \frac{u_{i-2}\,\omega'_{s_1,i-2}\,\omega'_{s_2,i-1}}{a_{i-1}} + \frac{u_i\,\omega'_{s_2,i}\,\omega'_{s_2,i-1}}{a_{i-1}} \\ &= \frac{1}{h^2}\left(\frac{u_{i+2}-u_i}{a_{i+1}} - \frac{u_{i-2}-u_i}{a_{i-1}}\right) \sim \delta_x\left(\frac{\delta_x u}{a}\right),\end{aligned}$$

where $\omega'_{s_p,j}$ denotes the derivative of the barycentric coordinate of the point j in the element s_p. We observe that this approximation is consistent with the classical scheme.

In two dimensions of space, the derivatives are simply written as:

$$\nabla \cdot \left(\frac{\nabla u}{a}\right) = \frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{a}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{a}\right),$$

and thus the approximation of the curvature at the grid point (i,j) becomes:

$$\begin{aligned}\nabla \cdot \left(\frac{\nabla u}{a}\right)_{i,j} &= \frac{\left(\frac{\frac{\partial u}{\partial x}}{a}\right)_{i+\frac{1}{2},j} - \left(\frac{\frac{\partial u}{\partial x}}{a}\right)_{i-\frac{1}{2},j}}{\Delta x} + \frac{\left(\frac{\frac{\partial u}{\partial y}}{a}\right)_{i,j+\frac{1}{2}} - \left(\frac{\frac{\partial u}{\partial y}}{a}\right)_{i,j-\frac{1}{2}}}{\Delta y} \\ &= \frac{1}{h^2}\left(\frac{u_{i+1,j}-u_{i,j}}{a_{i+\frac{1}{2},j}} - \frac{u_{i,j}-u_{i-1,j}}{a_{i-\frac{1}{2},j}} + \frac{u_{i,j+1}-u_{i,j}}{a_{i,j+\frac{1}{2}}} - \frac{u_{i,j}-u_{i,j-1}}{a_{i,j-\frac{1}{2}}}\right).\end{aligned}$$

Like we did previously with the Laplacian, we split now the analysis of our scheme into several steps, and we consider namely:

- the coefficients of the principal cross in the stencil:

$$\frac{1}{64h^2}\begin{bmatrix} & & 4 & & \\ & & 0 & & \\ 4 & 0 & -16 & 0 & 4 \\ & & 0 & & \\ & & 4 & & \end{bmatrix}$$

  that leads to:

$$\begin{aligned}&\frac{1}{64h^2}\left(\frac{4u_{i,j+2}-4u_{i,j}}{a_{i,j+1}} - \frac{4u_{i,j}-4u_{i,j-2}}{a_{i,j-1}} + \frac{4u_{i+2,j}-4u_{i,j}}{a_{i+1,j}} - \frac{4u_{i,j}-4u_{i-2,j}}{a_{i-1,j}}\right) \\ =&\frac{4}{64}\times\frac{1}{(2h)^2}\left(\frac{u_{i,j+2}-u_{i,j}}{a_{i,j+1}} - \frac{u_{i,j}-u_{i,j-2}}{a_{i,j-1}} + \frac{u_{i+2,j}-u_{i,j}}{a_{i+1,j}} - \frac{u_{i,j}-u_{i-2,j}}{a_{i-1,j}}\right) \sim \frac{1}{4}\nabla\cdot\left(\frac{\nabla u}{a}\right),\end{aligned} \tag{15}$$

- the coefficients of the second and fourth columns of the stencil:

$$\frac{1}{64h^2}\begin{bmatrix} & 4 & & & \\ & 0 & & & \\ & -8 & & & \\ & 0 & & & \\ & 4 & & & \end{bmatrix} \quad \text{and} \quad \frac{1}{64h^2}\begin{bmatrix} & & & 4 & \\ & & & 0 & \\ & & & -8 & \\ & & & 0 & \\ & & & 4 & \end{bmatrix}$$

  leading respectively to:

$$\begin{aligned}&\frac{1}{64h^2}\left(\frac{2u_{i-1,j+2}-2u_{i-1,j}}{a_{i,j+1}} - \frac{2u_{i-1,j}-2u_{i-1,j-2}}{a_{i,j-1}} + \frac{2u_{i-1,j+2}-2u_{i-1,j}}{a_{i-1,j-1}} - \frac{2u_{i-1,j}-2u_{i-1,j+2}}{a_{i-1,j+1}}\right) \text{ and} \\ &\frac{1}{64h^2}\left(\frac{2u_{i+1,j+2}-2u_{i+1,j}}{a_{i,j+1}} - \frac{2u_{i+1,j}-2u_{i+1,j-2}}{a_{i,j-1}} + \frac{2u_{i+1,j+2}-2u_{i+1,j}}{a_{i+1,j+1}} - \frac{2u_{i+1,j}-2u_{i+1,j-2}}{a_{i+1,j-1}}\right),\end{aligned} \tag{16}$$

- the coefficients of the second and fourth raws:

$$\frac{1}{64h^2}\begin{bmatrix} & & & & \\ 4 & 0 & -8 & 0 & 4 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \quad \text{and} \quad \frac{1}{64h^2}\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ 4 & 0 & -8 & 0 & 4 \\ & & & & \end{bmatrix}$$

leading respectively to:

$$\frac{1}{64h^2}\left(\frac{2u_{i+2,j+1}-2u_{i,j+1}}{a_{i+1,j}}-\frac{2u_{i,j+1}-2u_{i-2,j+1}}{a_{i-1,j}}-\frac{2u_{i+2,j+1}-2u_{i,j+1}}{a_{i+1,j+1}}-\frac{2u_{i,j+1}-2u_{i-2,j+1}}{a_{i-1,j+1}}\right) \text{ and}$$
$$\frac{1}{64h^2}\left(\frac{2u_{i+2,j-1}-2u_{i,j-1}}{a_{i+1,j}}-\frac{2u_{i,j-1}-2u_{i-2,j-1}}{a_{i-1,j}}+\frac{2u_{i+2,j-1}-2u_{i,j-1}}{a_{i+1,j-1}}-\frac{2u_{i,j-1}-2u_{i-2,j-1}}{a_{i-1,j-1}}\right). \tag{17}$$

One can easily observe that: (16) + (17) $\sim \frac{1}{2}\nabla\cdot\left(\frac{\nabla u}{a}\right)$.

- the remaining coefficients:

$$\frac{1}{64h^2}\begin{bmatrix} 2 & & & & 2 \\ & 0 & & 0 & \\ & & -8 & & \\ & 0 & & 0 & \\ 2 & & & & 2 \end{bmatrix}$$

corresponding to:

$$\frac{1}{64h^2}\left(\frac{2u_{i+2,j-2}-2u_{i,j}}{a_{i+1,j-1}}-\frac{2u_{i,j}-2u_{i+2,j+2}}{a_{i+1,j+1}}+\frac{2u_{i-2,j+2}-2u_{i,j}}{a_{i-1,j+1}}-\frac{2u_{i,j}-2u_{i-2,j-2}}{a_{i-1,j-1}}\right) \tag{18}$$
$$\sim \frac{1}{4}\nabla\cdot\left(\frac{\nabla u}{a}\right).$$

Then, by summing (15), (16), (17) and (18), we deduce that our scheme is consistent with the operator $\nabla\cdot\left(\frac{\nabla}{a}\right)$.

In this section, we have shown that our schemes are both consistent with the differential operators of the model Equation (2). These results have been obtained on Cartesian grids for clarity purpose. Moreover, there is no canonical discretization of the Laplacian (or the general curvature term) on arbitray triangulations. Nonetheless, there is no obvious reason for which the consistency would not be verified in such configuration. In the next section, we will establish a stability result for the numerical schemes.

4. Stability issues

In this section, the stability of the numerical schemes, with respect to the L^2-norm, is established. Indeed, we found that none of the schemes is stable for the L^∞-norm. This may not be a drawback however, since all calculations are based on triangulations. In the type of application envisaged, the stopping criterion in the algorithm is based on the L^2-norm and it seems thus more pertinent to use this norm for showing the stability of the numerical schemes.

We present the L^2-stability for the explicit and semi-implicit schemes, like previously, first by considering the case of the Laplace operator and then by dealing with the general mean curvature case. A general stability criterion for the time step (CFL condition) is given on any type of mesh and the constant involved in this condition is explicited in the case of Cartesian grids.

4.1. The first order explicit scheme

We recall that the first order explicit scheme we consider for solving the Equation (2) is defined as follows:

$$u_i^{n+1} = u_i^n + \Delta t\, d_i\left(\left(\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\right)_i^n + \alpha\right). \tag{19}$$

By introducing the discretization matrix M of the operator $\nabla \cdot \left(\dfrac{\nabla}{|\nabla|}\right)$ and D the diagonal matrix with d_i as diagonal coefficients, Equation (19) becomes:

$$u_i^{n+1} = ((I + \Delta t\, D\, M)\, u)_i^n + \Delta t\, d_i\, \alpha.$$

4.1.1. *Laplacian case*

For the sake of understanding, we consider first the case of the Laplacian. The L^2-stability property is equivalent to showing that $||I + \Delta t\, D\, M||_2 \leq 1$. To this end, we write the L^2-norm of $I + \Delta t\, D\, M$ as follows:

$$||I + \Delta t\, D\, M||_2 = ||D\, \Lambda^{-1}(\Lambda\, D^{-1} + \Delta t\, \Lambda\, M)||_2$$

where Λ is a diagonal matrix such that $(\Lambda)_{i,i} = \sum_{K \in \mathcal{B}_i} A_K$ and thus we obtain:

$$||I + \Delta t\, D\, M||_2 \quad \leq \quad ||D\, \Lambda^{-1}||_2\, ||\Lambda\, D^{-1} + \Delta t\, \Lambda\, M||_2.$$

In order to ensure $||I + \Delta t\, D\, M||_2 < 1$, it is sufficient to show that $||\Lambda\, D^{-1} + \Delta t\, \Lambda\, M||_2 \leq \dfrac{1}{||D\, \Lambda^{-1}||_2}$. At first, we show that the matrix ΛM is symmetric, *i.e.* that: $\langle \Lambda\, M\, u, v\rangle = \langle u, \Lambda\, M\, v\rangle$, for all u and for all v. Indeed, using (10), we have:

$$(\Delta u)_i = \frac{\displaystyle\sum_{K \in \mathcal{B}_i} A_K \sum_{i_2 \in K} \langle (\nabla u)_{i_2}, \nabla \omega_{K,i_2}\rangle}{\displaystyle\sum_{K \in \mathcal{B}_i} A_K} \qquad \text{with} \qquad (\nabla u)_{i_2} = \frac{\displaystyle\sum_{L \in \mathcal{B}_{i_2}} A_L \sum_{i_3 \in L} u_{i_3} \nabla \omega_{L,i_3}}{\displaystyle\sum_{L \in \mathcal{B}_{i_2}} A_L},$$

and thus, we write:

$$\begin{aligned}
\langle \Lambda\, M\, u, v\rangle &= \sum_i (\Lambda\, M\, u)_i\, v_i = \sum_i v_i \sum_{K \in \mathcal{B}_i} A_K\, (\Delta u)_i \\
&= \sum_i v_i \sum_{K \in \mathcal{B}_i} A_K \sum_{i_2 \in K} \langle (\nabla u)_{i_2}, \nabla \omega_{K,i_2}\rangle \\
&= \sum_i \sum_{K \in \mathcal{B}_i} \sum_{i_2 \in K} v_i A_K \langle (\nabla u)_{i_2}, \nabla \omega_{K,i_2}\rangle.
\end{aligned}$$

Using a renumbering procedure, ($K \in \mathcal{B}_i \Rightarrow i \in K$) and ($i_2 \in K \Rightarrow K \in \mathcal{B}_{i_2}$) we obtain :

$$\langle \Lambda\, M\, u, v\rangle = \sum_{i_2} \langle (\nabla u)_{i_2}, \sum_{K \in \mathcal{B}_{i_2}} A_K \sum_{i \in K} v_i \nabla \omega_{K,i_2}\rangle.$$

The next step in order to have $\Lambda\, M$ symmetric is to show that, for all i_2:

$$\sum_{K \in \mathcal{B}_{i_2}} A_K \sum_{i \in K} u_i\, \nabla \omega_{K i_2} = - \sum_{K \in \mathcal{B}_{i_2}} A_K \sum_{i \in K} u_i\, \nabla \omega_{K i}$$

i.e., for all i_2:

$$\sum_{K \in \mathcal{B}_{i_2}} A_K \sum_{i \in K} u_i (\nabla \omega_{K i_2} + \nabla \omega_{K i}) = 0 \quad \Leftrightarrow \quad \sum_{i \in \mathcal{B}_{i_2}} \sum_{K \in \mathcal{B}_{i_2} \cap \mathcal{B}_i} u_i\, A_K (\nabla \omega_{K i_2} + \nabla \omega_{K i}) = 0. \tag{20}$$

It is sufficient to show that for all i_2, $\displaystyle\sum_{K \in \mathcal{B}_{i_2} \cap \mathcal{B}_i} A_K (\nabla \omega_{K i_2} + \nabla \omega_{K i}) = 0,\ \forall i \in \mathcal{B}_{i_2}$. This condition is also a necessary condition, for the equality (20) to be satisfied for all u_i. We deal with two cases:

- if $i = i_2$, the same term $\nabla\omega_{Ki_2}$ appears twice in the previous sum. We define $\omega = \sum_{K\in\mathcal{B}_{i_2}} \omega_{Ki_2}\mathbb{I}_K$, and thus we have: $\sum_{K\in\mathcal{B}_{i_2}} A_K \nabla\omega_{Ki_2} = \int_{\cup_{K\in\mathcal{B}_{i_2}} K} \nabla\omega = \int_{\partial(\cup K)} \omega \cdot \vec{n} = 0$, this term is vanishing since $w = 0$ on $\partial(\cup K)$. And hence the result follows:

$$\sum_{K\in\mathcal{B}_{i_2}} A_K\, \nabla\omega_{Ki_2} = 0,$$

- if $i \neq i_2$, the sum becomes:

$$\sum_{K\in\mathcal{B}_i\cap\mathcal{B}_{i_2}} A_K(\nabla\omega_{Ki_2} + \nabla\omega_{Ki}) = A_K(\nabla\omega_{Ki_2} + \nabla\omega_{Ki}) + A_L(\nabla\omega_{Li_2} + \nabla\omega_{Li}),$$

  where K and L denote the two triangles sharing the edge (i, i_2), Figure 3.

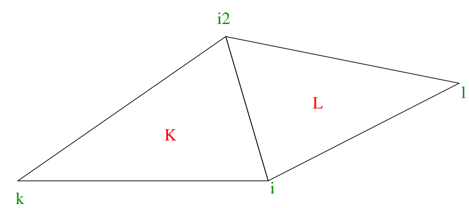

FIGURE 3. The configuration of two triangles sharing the edge (i, i_2)

  Since $\sum_{i_2\in K} \nabla\omega_{Ki_2} = 0$ for all i_2, then $\nabla\omega_{Ki_2} + \nabla\omega_{Ki} = -\nabla\omega_{Kk}$ and $\nabla\omega_{Li_2} + \nabla\omega_{Li} = -\nabla\omega_{Ll}$. Considering the two points opposite to the edge (i, i_2) in K and L, we like to show that:

$$A_K\nabla\omega_{Kk} = -A_L\nabla\omega_{Ll}.$$

  We notice that $\nabla\omega_{Kk}$ is orthogonal to the edge (i, i_2), as well as the term $\nabla\omega_{Ll}$. Moreover, $|\nabla\omega_{Kk}| = \dfrac{1}{h_k}$ where h_k is the height of K emanating from k and $A_K = \dfrac{h_k}{2} \times |(i, i_2)|$, where $|(i, i_2)|$ denotes the length of the edge (i, i_2), we conclude that $|A_K\nabla\omega_{Kk}| = \dfrac{|(i, i_2)|}{2}$. Likewise, on the triangle L, we obtain a similar relation: $|A_L\nabla\omega_{Ll}| = \dfrac{|(i, i_2)|}{2}$. Notice that the vectors $\nabla\omega_{Kk}$ and $\nabla\omega_{Ll}$ have the same direction but opposite sign. Hence, $A_K\, \nabla\omega_{Kk} = -A_L\, \nabla\omega_{Ll}$ and the results (20) follows.

We have just shown that:

$$\sum_{K\in\mathcal{B}_{i_2}} A_K \sum_{i\in K} u_i\, \nabla\omega_{Ki_2} = -\sum_{K\in\mathcal{B}_{i_2}} A_K \sum_{i\in K} u_i\, \nabla\omega_{Ki}. \tag{21}$$

On the other hand, we have the following equality:

$$\sum_{K\in\mathcal{B}_{i_2}} A_K \sum_{i\in K} v_i\, \nabla\omega_{K,i_2} = -\sum_{K\in\mathcal{B}_{i_2}} A_K\, (\nabla v)_{i_2},$$

and then, we obtain the identity:

$$\langle \Lambda M u, v\rangle = -\sum_{i_2} \langle (\nabla u)_{i_2}, (\nabla v)_{i_2}\rangle \sum_{K\in\mathcal{B}_{i_2}} A_K. \tag{22}$$

Now, we have to compare $\langle \Lambda M u, v\rangle$ with $\langle u, \Lambda M v\rangle$. Replacing u by v and v by u in (22), leads to write:

$$\langle u, \Lambda M v\rangle = \sum_{i_2} \langle (\nabla v)_{i_2}, (\nabla u)_{i_2}\rangle \sum_{K\in\mathcal{B}_{i_2}} A_K = \langle \Lambda M u, v\rangle.$$

Hence, the matrix ΛM is symmetric. Consequently, the matrix $\Lambda\, D^{-1} + \Delta t\, \Lambda\, M$ is also symmetric and, in order to have $||I + \Delta t\, D\, M||_2 \leq 1$, it is sufficient to show that:

$$\rho(\Lambda\, D^{-1} + \Delta t\, \Lambda\, M) \quad \leq \quad \frac{1}{||D\,\Lambda^{-1}||_2} = \frac{1}{\rho(D\,\Lambda^{-1})}, \tag{23}$$

where $\rho(P)$ is the spectral radius of the matrix P.

In a second step, we like to show that the inequality (23) holds and then, that all the eigenvalues of M are negative. To this end, we consider the simplified parabolic model $\partial_t u = \Delta u$. We have then:

$$u\, \partial_t u = u\, \Delta u \quad \Leftrightarrow \quad \partial_t \frac{u^2}{2} = u\, \Delta u \quad \Leftrightarrow \quad \partial_t \int \frac{u^2}{2} = \int u\, \Delta u,$$

that leads to the following inequality:

$$\int u\, \Delta u = -\int |\nabla u|^2 \leq 0. \tag{24}$$

Hence, we have to show that the numerical scheme satisfies such an inequality, for all eigenvalues to be negative. Using the previous formulas given in Section 2, we write:

$$\sum_i u_i\, (\Delta u)_i = \sum_i u_i \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K \sum_{i_2\in K} \langle (\nabla u)_{i_2}, \nabla\omega_{K i_2}\rangle}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K},$$

and thus we have:

$$\sum_i \left(\sum_{K\in\mathcal{B}_i} A_K\right) u_i\, (\Delta u)_i = \sum_i \sum_{K\in\mathcal{B}_i} \sum_{i_2\in K} u_i\, A_K\, \langle (\nabla u)_{i_2}, \nabla\omega_{K i_2}\rangle = \sum_{\substack{i,K,i_2 \\ K\in\mathcal{B}_i, i_2\in K}} u_i\, A_K \langle (\nabla u)_{i_2}, \nabla\omega_{K i_2}\rangle.$$

Using a renumbering procedure, ($K \in \mathcal{B}_i \Rightarrow i \in K$) and ($i_2 \in K \Rightarrow K \in \mathcal{B}_{i_2}$), we obtain:

$$\sum_i \left(\sum_{K\in\mathcal{B}_i} A_K\right) u_i\, (\Delta u)_i = \sum_{\substack{i,K,i_2 \\ K\in\mathcal{B}_{i_2}, i\in K}} u_i\, A_K \langle (\nabla u)_{i_2}, \nabla\omega_{K i_2}\rangle = \sum_{i_2} \sum_{K\in\mathcal{B}_{i_2}} \sum_{i\in K} u_i\, A_K \langle (\nabla u)_{i_2}, \nabla\omega_{K i_2}\rangle,$$

and thus:

$$\sum_i \left(\sum_{K\in\mathcal{B}_i} A_K\right) u_i\, (\Delta u)_i = \sum_{i_2} \langle (\nabla u)_{i_2}, \sum_{K\in\mathcal{B}_{i_2}} A_K \sum_{i\in K} u_i\, \nabla\omega_{K i_2}\rangle. \tag{25}$$

Using the identity (21), we have:

$$\sum_{K\in\mathcal{B}_{i_2}} A_K \sum_{i\in K} u_i\, \nabla\omega_{K i_2} = -(\nabla u)_{i_2} \sum_{K\in\mathcal{B}_{i_2}} \sum_{i\in K} A_K.$$

Hence, combining the last two results leads to a *discrete* Green formula:

$$\sum_i \left(\sum_{K\in\mathcal{B}_i} A_K \right) u_i\,(\Delta u)_i = -\sum_{i_2} \left(\sum_{K\in\mathcal{B}_{i_2}} \sum_{i\in K} A_K \right) |(\nabla u)_{i_2}|^2 \tag{26}$$

in which, obviously, the right-hand side term is negative. Given the Laplacian matrix M, we have shown that:

$$\sum_i \left(\sum_{K\in\mathcal{B}_i} A_K \right) u_i\,(Mu)_i \leq 0.$$

Introducing the notation $\lambda_i = \sum\limits_{K\in\mathcal{B}_i} A_K$, yields: $\sum\limits_i \sum\limits_{K\in\mathcal{B}_i} A_K\, u_i\,(\Delta u)_i \leq 0 \Leftrightarrow \sum\limits_i \lambda_i\, u_i\,(Mu)_i \leq 0$ and consequently $\sum\limits_i (\sqrt{\lambda_i}\, u_i)(\sqrt{\lambda_i}\,(Mu)_i) \leq 0$. Defining the diagonal matrix Λ with λ_i as diagonal coefficients, leads to rewrite the inequality as follows:

$$\begin{aligned}
\sum_i (\sqrt{\Lambda}\,u)_i(\sqrt{\Lambda}\,Mu)_i \leq 0 \quad &\Leftrightarrow \quad \sum_i (\sqrt{\Lambda}\,u)_i((\sqrt{\Lambda}\,M\sqrt{\Lambda}^{-1})\sqrt{\Lambda}\,u)_i \leq 0 \\
&\Leftrightarrow \quad \sum_i (\sqrt{\Lambda}\,u)_i(B\sqrt{\Lambda}\,u)_i \leq 0,
\end{aligned}$$

where $B = \sqrt{\Lambda}\,M\sqrt{\Lambda}^{-1}$. Let $v \in \mathbb{R}^N$ be such that $u = \sqrt{\Lambda}^{-1}v$, then $\sum\limits_i v_i(Bv)_i \leq 0$, and thus, B is a semi-negative definite matrix. Since the spectrum $\mathrm{Sp}(M) = \mathrm{Sp}(B)$, M is also a semi-negative definite matrix. We have then $u^T Mu \leq 0$ for all u and if (λ, u) are pairs of eigenvalues and eigenvectors of M, then $u^T Mu = u^T \lambda u = \lambda u^T u \leq 0$. However, since $u^T u \geq 0$, then $\lambda \leq 0$ and thus : $\mathrm{Sp}(M) \leq 0$. Moreover, this result directly implies the inequality (23). In order to have the stability of the scheme, we have to satisfy a condition on the time step, *i.e.* we look for Δt such that:

$$\rho(\Lambda\, D^{-1} + \Delta t\,\Lambda\, M) \quad \geq \quad -\frac{1}{||D\,\Lambda^{-1}||_2} = -\rho(\Lambda\, D^{-1}).$$

Hence, the expected value of Δt is such that:

$$\begin{aligned}
\rho(\Lambda\, D^{-1} + \Delta t\,\Lambda\, M) + \rho(\Lambda\, D^{-1}) \geq 0 \quad &\Leftrightarrow \quad \rho(2\Lambda\, D^{-1} + \Delta t\,\Lambda\, M) \geq 0 \\
&\Leftrightarrow \quad u^T(2\Lambda\, D^{-1} + \Delta t\,\Lambda\, M)\,u \geq 0 \quad \forall u \\
&\Leftrightarrow \quad \sum_i u_i((2\Lambda\, D^{-1} + \Delta t\,\Lambda\, M)\,u)_i \geq 0 \\
&\Leftrightarrow \quad \Delta t \sum_i u_i(\Lambda\, M\, u)_i \geq -2\sum_i u_i(\Lambda\, D^{-1}u)_i.
\end{aligned}$$

Using identity (22), we deduce the following stability criterion for the time step:

$$\Delta t \sum_i |(\nabla u)_i|^2\,\lambda_i \quad \leq \quad 2\sum_i \frac{\lambda_i}{d_i}\,u_i^2. \tag{27}$$

In order to give a explicit expression for this criterion, on Cartesian grids, we need to find now an upper bound for $\sum_i |(\nabla u)_i|^2\,\lambda_i$. To this end, according to (6), we recall that:

$$(\nabla u)_i = \frac{\sum\limits_{K\in\mathcal{B}_i} A_K \sum\limits_{i_2\in K} u_{i_2}\nabla\omega_{K,i_2}}{\sum\limits_{K\in\mathcal{B}_i} A_K}.$$

And thus, we deduce that:

$$|(\nabla u)_i|^2 \quad \leq \quad \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K^2 |\sum_{i_2\in K} u_{i_2}\nabla\omega_{K,i_2}|^2}{\left(\displaystyle\sum_{K\in\mathcal{B}_i} A_K\right)^2}.$$

Consequently, we have:

$$\begin{aligned}
\lambda_i|(\nabla u)_i|^2 = \sum_{K\in\mathcal{B}_i} A_K|(\nabla u)_i| \quad &\leq \quad \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K^2 |\sum_{i_2\in K} u_{i_2}\nabla\omega_{K,i_2}|^2}{\sum_{K\in\mathcal{B}_i} A_K} \\
&\leq \quad \max_{K,i_2}|\nabla\omega_{K,i_2}|^2 \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K^2 \left(\sum_{i_2\in K} u_{i_2}\right)^2}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} \\
&\leq \quad n\,\max_{K,i_2}|\nabla\omega_{K,i_2}|^2 \frac{\displaystyle\sum_{K\in\mathcal{B}_i} A_K^2 \sum_{i_2\in K} u_{i_2}^2}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K},
\end{aligned}$$

where n denotes the number of vertices in a cell (here $n = 4$). Hence, we obtain, by summing on the index i:

$$\sum_i \lambda_i|(\nabla u)_i|^2 \quad \leq \quad n\,\max_{K,i_2}|\nabla\omega_{K,i_2}|^2 \frac{\displaystyle\sum_i\sum_{K\in\mathcal{B}_i} A_K^2 \sum_{i_2\in K} u_{i_2}^2}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K}.$$

Using a renumbering procedure: ($K \in \mathcal{B}_i \Rightarrow i \in K$) and ($i_2 \in K \Rightarrow K \in \mathcal{B}_{i_2}$), we write then:

$$\frac{\displaystyle\sum_i\sum_{K\in\mathcal{B}_i} A_K^2 \sum_{i_2\in K} u_{i_2}^2}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} = \sum_{i_2} u_{i_2}^2 \frac{\displaystyle\sum_{K\in\mathcal{B}_{i_2}}\sum_{i\in K} A_K^2}{\displaystyle\sum_{K\in\mathcal{B}_{i_2}} A_K}.$$

On Cartesian grids, if h denotes the size of the cells, we have: $\dfrac{\displaystyle\sum_i\sum_{K\in\mathcal{B}_i} A_K^2 \sum_{i_2\in K} u_{i_2}^2}{\displaystyle\sum_{K\in\mathcal{B}_i} A_K} \leq 4h^2 \displaystyle\sum_{i_2} u_{i_2}^2$ and thus we obtain the following estimate:

$$\sum_i \lambda_i|(\nabla u)_i|^2 \quad \leq \quad 4\max_{K,i_2}|\nabla\omega_{K,i_2}|^2 \times 4h^2 \sum_{i_2} u_{i_2}^2.$$

We recall that $|\nabla\omega_{K,i_2}|^2 = \dfrac{1}{2h^2}$ and then the time step Δt must be such that:

$$8\sum_{i_2} u_{i_2}^2 \quad \leq \quad \frac{2}{\Delta t}\sum_i \frac{\lambda_i}{d_i} u_i^2 \quad \leq \quad \frac{2}{\Delta t}\sum_i \frac{1}{d_i}\left(\sum_{K\in\mathcal{B}_i} A_K\right) u_i^2 \quad \leq \quad \frac{2}{\Delta t}\sum_i \frac{4h^2}{d_i} u_i^2.$$

Then, the stability (CFL) condition consists in writting that Δt shall be such that:

$$\max_i d_i \; 8 \sum_{i_2} u_{i_2}^2 \quad \leq \quad \frac{2}{\Delta t} \, 4h^2 \sum_i u_i^2.$$

Hence, the CFL condition to ensure the stability of the explicit scheme is finally:

$$8\beta \leq \frac{2}{\Delta t} 4h^2 \quad \Leftrightarrow \quad \Delta t \leq \frac{h^2}{\beta},$$

where $\beta = \max_i d_i$. In the applications envisaged, the coefficients β is bounded since the domain is an open bounded set.

4.1.2. *General mean curvature case*

By analogy with the Laplacian case, we consider here the matrix M of the operator $\nabla \cdot \left(\frac{\nabla}{|\nabla|} \right)$. Like previously, we need to show that $||I + \Delta t \, D \, M||_2 \leq 1$. We use the same steps as previously that is, first we have to show the symmetry of the matrix ΛM. Using Equation (9) we recall that:

$$\left(\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \right)_i = \frac{\displaystyle\sum_{K \in \mathcal{B}_i} A_K \sum_{i_2 \in K} \langle \frac{(\nabla u)_{i_2}}{|\nabla u|_{i_2}}, \nabla \omega_{K,i_2} \rangle}{\displaystyle\sum_{K \in \mathcal{B}_i} A_K}$$

with $|\nabla u|_{i_2} = |(\nabla u)_{i_2}|$. Let $\alpha_{i_2} = |\nabla u|_{i_2}$ then, if $\alpha_{i_2} \neq 0$:

$$\left(\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \right)_i = \frac{\displaystyle\sum_{K \in \mathcal{B}_i} A_K \sum_{i_2 \in K} \langle \frac{(\nabla u)_{i_2}}{\alpha_{i_2}}, \nabla \omega_{K,i_2} \rangle}{\displaystyle\sum_{K \in \mathcal{B}_i} A_K}.$$

Like previously we show the symmetry of the matrix $\Lambda \, M$ as follows:

$$\langle \Lambda \, Mu, v \rangle = - \sum_{i_2} \langle (\nabla u)_{i_2}, (\nabla v)_{i_2} \rangle \frac{\displaystyle\sum_{K \in \mathcal{B}_{i_2}} A_K}{\alpha_{i_2}} = \langle u, \Lambda \, Mv \rangle.$$

To show the L^2-stability of our scheme, we proceed as for the Laplacian case, the restriction on the time step comes here from the inequality: $\rho(2\Lambda \, D^{-1} + \Delta t \, \Lambda \, M) \geq 0$.

This strong numerical restriction on the time step reinforces even more the need for a semi-implicit time-stepping scheme on unstructured triangulations.

4.2. **First order semi-implicit scheme**

We recall that the first order semi-implicit scheme, for the Equation (2), we use, is defined as follows:

$$u_i^{n+1} - d_i \, \Delta t \left(\nabla \cdot \left(\frac{\nabla u^{n+1}}{|\nabla u^n|} \right) \right)_i = d_i \, \Delta t \, \alpha + u_i^n. \tag{28}$$

Introducing the discretization matrix M of the operator $\nabla \cdot \left(\frac{\nabla}{|\nabla|} \right)$, Equation (28) becomes:

$$((I - \Delta t \, DM)u)_i^{n+1} = u_i^n + \Delta t \, d_i \, \alpha.$$

At first, we show that the matrix $(I - \Delta t\, D\, M)$ is invertible. We have already established that $\mathrm{Sp}(M) \leq 0$, and thus $\mathrm{Sp}(\Lambda M) \leq 0$.

To this end, we want to show that $\mathrm{Ker}(I - \Delta t\, D\, M) = 0$. We suppose that there exists $u \neq 0$, such that $(I - \Delta t\, D\, M)u = 0$. Introducing the diagonal matrix Λ, we have $(I - \Delta t\, D\, \Lambda^{-1}\, \Lambda\, M)u = 0$, then $(D\, \Lambda^{-1} - \Delta t\, \Lambda\, M)u = 0$. Hence, $u^T(D\, \Lambda^{-1} - \Delta t\, \Lambda\, M)u = 0$ and $u^T\, D\, \Lambda^{-1}u = \Delta t\, u^T \Lambda\, M\, u$, which is impossible because $u^T\, D\, \Lambda^{-1}u > 0$, if $u \neq 0$ and $\Lambda\, M$ is a symmetric negative semi-definite matrix. Thus, we conclude that $\mathrm{Ker}(I - \Delta t\, D\, M) = 0$ and that $(I - \Delta t\, D\, M)$ is invertible.

We have now to show that $||(I - \Delta t\, D\, M)^{-1}||_2 \leq 1 \Leftrightarrow ||I - \Delta t\, D\, M||_2 \geq 1$. For all u, we have:

$$\begin{aligned}
||(I - \Delta t\, D\, M)u||_2^2 &= u^T(I - \Delta t\, D\, M)^T(I - \Delta t\, D\, M)u \\
&= u^T(I - \Delta t\, M^T\, D)(I - \Delta t\, D\, M)u \\
&= u^T u - \Delta t\, u^T(M^T\, D + D\, M)u + \Delta t^2\, u^T M^T\, D\, D\, M u \\
&\geq u^T u - \Delta t\, u^T(M^T\, D + D\, M)u
\end{aligned}$$

as $u^T\, M^T\, D\, D\, M u = (D\, M u)^T\, (D\, M u) \geq 0$. Next, we show that $u^T(M^T\, D + D\, M)u \leq 0$. The matrix $\Lambda\, M$ being symmetric negative semi-definite, we have:

$$\begin{aligned}
u^T(M^T D + D\, M)u &= u^T((\Lambda M)^T\, \Lambda^{-1}\, D + D\, \Lambda^{-1}\, \Lambda\, M)u = u^T(\Lambda M\, \Lambda^{-1}\, D + D\, \Lambda^{-1}\, \Lambda\, M)u \\
&= (\sqrt{D\, \Lambda^{-1}}u)^T(\sqrt{D\, \Lambda^{-1}})^{-1}\, \Lambda\, M\, (\sqrt{D\, \Lambda^{-1}})(\sqrt{D\, \Lambda^{-1}}u) \\
&\quad + (\sqrt{D\, \Lambda^{-1}}u)^T(\sqrt{D\, \Lambda^{-1}})\, \Lambda\, M\, (\sqrt{D\, \Lambda^{-1}})^{-1}(\sqrt{D\, \Lambda^{-1}}u) \\
&= (\sqrt{D\, \Lambda^{-1}}u)^T\, ((\sqrt{D\, \Lambda^{-1}})^{-1}\, \Lambda\, M\, (\sqrt{D\, \Lambda^{-1}}) + (\sqrt{D\, \Lambda^{-1}})\, \Lambda\, M\, (\sqrt{D\, \Lambda^{-1}})^{-1})\, (\sqrt{D\, \Lambda^{-1}}u).
\end{aligned}$$

We denote $N = (\sqrt{D\, \Lambda^{-1}})^{-1}\, \Lambda\, M\, (\sqrt{D\, \Lambda^{-1}}) + (\sqrt{D\, \Lambda^{-1}})\, \Lambda\, M\, (\sqrt{D\, \Lambda^{-1}})^{-1}$. The matrix N is symmetric, since the matrices $\Lambda\, M$ and $(\sqrt{D\, \Lambda^{-1}})$ are symmetric. Then $u^T(M^T\, D + D\, M)u \leq 0$ if and only if $\mathrm{Sp}(N) \leq 0$ but $\mathrm{Sp}(N) = 2 \times \mathrm{Sp}(\Lambda\, M) \leq 0$. We have then the expected result $||(I - \Delta t\, D\, M)^{-1}||_2 \leq 1$. It is important to notice here that this semi-explicit scheme is stable without any restrictive CFL condition on the time step (unconditionally stable).

These explicit and semi-implicit schemes are both consistent with the model Equation (2), cf. Section 3, and stable with respect to the L^2-norm. To emphasize their efficiency we propose an application example in the next section.

4.3. **Convergence**

In the linear case, that is, the Laplacian case, the Lax-Richtmyer theorem [19] can be directly applied to both our schemes. It states that a consistent scheme, for a well posed linear problem, is convergent if and only if it is stable. On the other hand, in the general mean curvature formulation, there is no equivalent result. The only work the authors are aware of is the paper of Barles and Souganidis, [3], that proves that any monotone, consistent and stable scheme converges in the context of fully non linear elliptic or parabolic second order equations provided that there exists a comparison principle for the limiting equation. In particular, this last point remains to be established in our case, to be able to invoke this result and to conclude about the convergence of our schemes.

5. Application to curve fitting

In this section, we present an experimental example of construction of a smooth curve approximating a given set of points V. This data set has been sampled from the analytical curve Γ corresponding to the implicit equation: $x^4 \times y^4 + \sin(4x) \times \sin(4y) - 0.8 = 0$.

We started from an initial sufficiently smooth level set function u defined over a square unit domain enclosing all points of V. For simplicity purposes, u was defined such that all level sets correspond to concentric circles, and we focus on the evolution of the zero level set of u, supposed to contain all points of V initially. The evolution problem was solved for the general Equation (2) (without assuming $|\nabla u| = 1$). We considered the numerical solution to be achieved (convergence of the algorithm) when the zero level set of u is sufficiently close to the points.

To illustrate the main features of our schemes, we performed the calculations on four triangular meshes representing the various connectivity and refinement types: two structured meshes of different mesh sizes M_1 and M_2, an unstructured uniform mesh M_3 and a mesh M_4 locally adapted to the zero level set of the distance function d to the data set (with a high density of elements in the vicinity of the points set [6]). Notice that in all meshes, the points of V are not mesh vertices and that the mesh size is here independent of the sampling of these points.

Table 1 reports the L^2 error err related to each mesh, computed as the absolute value of the level set function u at each point of V and expected to be close to zero. This value is obtained using an L^2 projection scheme on the current mesh. For the explicit scheme, we have prescribed a time step $dt = h^2$ as dictated by the most restrictive CFL condition. For the implicit scheme, the time step has been fixed as $dt = 10 \times h$. Here, the mesh size h corresponds to the minimal radius of the incircle of each triangle in the mesh. Finally, the number of vertices np and the number of iterations nit required to achieve convergence have been reported in this table.

	meshes	M_1	M_2	M_3	M_4
	np	7,773	27,133	8,964	17,275
	h	1.1×10^{-2}	5.5×10^{-3}	1.05×10^{-2}	1.06×10^{-3}
explicit	nit	150	340	160	1500
	err	1.01×10^{-2}	5.17×10^{-3}	9.36×10^{-3}	1.12×10^{-3}
implicit	nit	11	21	12	144
	err	8.13×10^{-3}	4.19×10^{-3}	8.43×10^{-3}	9.17×10^{-4}

Table 1. Statistics related to the convergence of the algorithms on different meshes.

From the table, we can conclude that the numerical results are in good accordance with the theoretical expectations and we can see that the L^2 error is of the order of the mesh size h. The numerical schemes scale well linearly as the number of mesh vertices increases. Moreover, the advantages of the implicit scheme over the explicit one are twofold: first, it converges much faster (the large time step doesn't impact the overall stability of the scheme) and second it is more accurate. In addition, we observe also that the parameters of the mesh (connectivity and density) have no influence on the convergence of these two algorithms.

Figure 4 shows the regular final curves (zero level set of u) obtained at convergence of the implicit scheme on the different meshes and a local enlargement to emphasize the efficiency of this scheme.

Additional examples of curve reconstruction on different points sets can be found in [6].

6. Conclusions

In this paper, we have proposed and analyzed two finite difference schemes for solving a PDE containing a mean curvature term on triangular meshes. The consistency and the stability of the schemes have been established both in the Laplacian case and in the general mean curvature formulation. An example of a curve reconstruction has been provided to illustrate the efficiency of the schemes on a points set corresponding to an analytical curve. This analysis can be naturally extended to the three dimensions and such schemes could be potentially applied to other Hamilton-Jacobi equations as well as to more general PDEs involving second order terms.

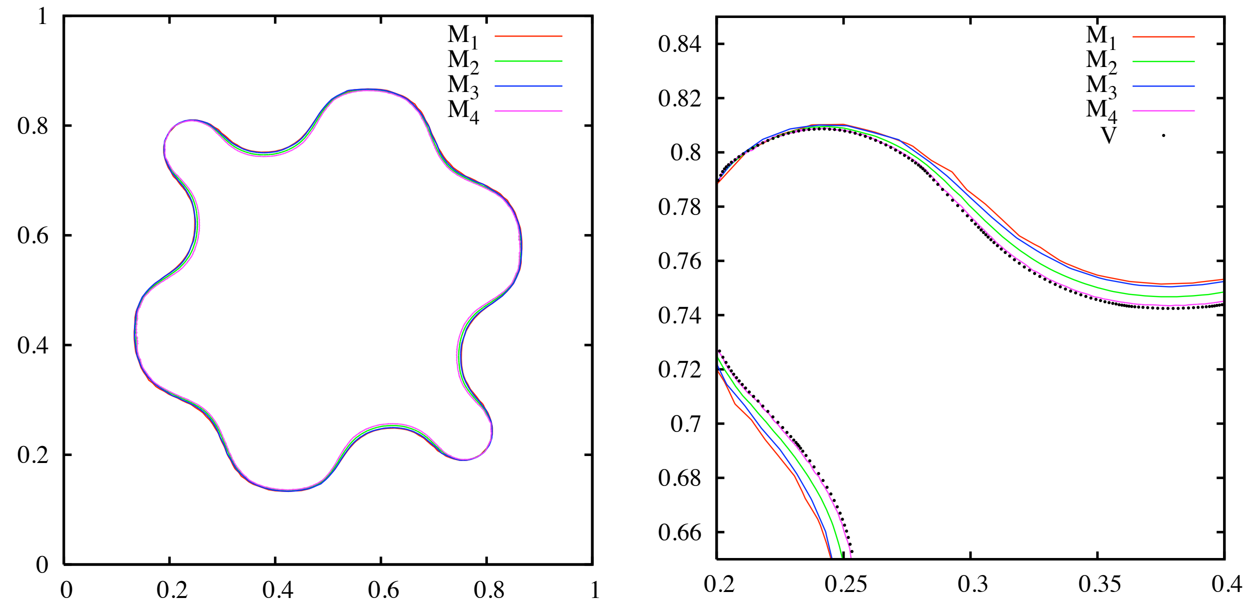

FIGURE 4. Curve reconstruction: zero level set of u at convergence on the four meshes (left hand side) and local enlargement (right hand side).

7. Acknowlegments

The authors acknowledge the valuable help of F. Lagoutière (University Denis Diderot, Paris 07) for countless and open discussions on the analysis of these schemes.

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